Iterative Methods for Solving linear systems of Algebraic Equation.

to solve the $n \times n$ linear system

$$AX = b$$

by iterative technique,

starts with an intial approximation

$$X^{(0)}$$

and generates a sequence of vectors

$$\{X^{(K)}\}_{K=0}^{\infty}$$

That Converge to x.

Consider the following linear n system of equations.

$$AX = b \qquad \text{"matrix form"}$$

which Can be written in form.

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{13} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \qquad (1)$$

(1) Jacobi's Method

let $x_i^{(1)}$ → iteration no

represent the Approximation vector of unknown

$x_i^{(1)}$ first iteration

$x_i^{(2)}$ second iteration.

$\vdots$

Then the Jacobi iterative method can be expressed as.

$$x_1^{n+1} = \frac{1}{a_{11}} \left(b_1 - a_{12} x_2^{n} - a_{13} x_3^{n} \right)$$

$$x_2^{n+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^n \quad - a_{23}x_3^n\right)$$

$$x_3^{n+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^n - a_{32}x_2^n \quad\right)$$

In general Case for m eqs.

$$\boxed{x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{m} a_{ij}\, x_j^{(n)}\right)}$$

Example

The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25,\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Sol

Solve eqs E_i for x_i

$$x_1^{n+1} = \frac{1}{10}\left(x_2^n - 2x_3^n \quad + 6\right)$$

$$x_2 = \frac{1}{11}\left(x_1^{n} + x_3^{n} + x_4^{n} + 25\right)$$

$$x_3^{n+1} = \frac{1}{10}\left(-2x_1^{n} + x_2^{n} + x_4^{n} + 11\right)$$

$$x_4^{n+1} = \frac{1}{8}\left(-3x_2^{n} + x_3^{n} + 15\right)$$

start with intial approximation $x_i^0 = (0,0,0,0)$ to get $x^{(1)}$

$$x_1^1 = \frac{1}{10}\left(x_2^0 - 2x_3^0 + 6\right) = 0.6000$$

$$x_2^1 = \frac{1}{11}\left(x_1^0 + x_3^0 + x_4^0 + 25\right) = 2.2727$$

$$x_3^1 = \frac{1}{10}\left(-2x_1^0 + x_2^0 + x_4^0 + 11\right) = -1.1000$$

$$x_4^1 = \frac{1}{8}\left(-3x_2^0 + x_3^0 + 15\right) = 1.875$$

additional iterates x^{n+1}, $n=0,\dots$, are generate in a Similar manner as presented in the following Table.

	0	1	2	3	4	5	6	7	8	9	10
x_1	0.000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
x_2	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
x_3	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
x_4	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

we stopped after 10 iterations because

$$\frac{\|x^{10}-x^{9}\|_\infty}{\|x^{10}\|_\infty} = \frac{8*10^{-4}}{1.9998} < 10^{-3}.$$

and

$$\|x^{10}-x\|_\infty = 0.0002.$$

---

② Gauss - Seidel method.

In this method the $(n+1)^{th}$ iterative values are used as soon as they are available and the iteration corresponding to eq (1) is defined by.

$$x_1^{n+1} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{n} - a_{13}x_3^{n}\right)$$

$$x_2^{n+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{23}x_3^{n}\right)$$

$$x_3^{n+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}\right)$$

in general case for m eqs.

$$\boxed{x_i^{n+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{n+1} - \sum_{j=i+1}^{m} a_{ij}x_j^{n}\right)}$$

$i = 1(1)m.$

Example 2

Use The Gauss-Seidel iteration method to Solve The following system of eqs.

The linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned}
E_1: &\quad 10x_1 - x_2 + 2x_3 = 6,\\
E_2: &\quad -x_1 + 11x_2 - x_3 + 3x_4 = 25,\\
E_3: &\quad 2x_1 - x_2 + 10x_3 - x_4 = -11,\\
E_4: &\quad 3x_2 - x_3 + 8x_4 = 15
\end{aligned}$$

has the unique solution $\mathbf{x} = (1, 2, -1, 1)^t$. Use Jacobi's iterative technique to find approximations $\mathbf{x}^{(k)}$ to $\mathbf{x}$ starting with $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$ until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}.$$

Sol Solve eqs E_i for x_i

$$x_1^{n+1} = \frac{1}{10}\left(x_2^{n} - 2x_3^{n} + 6 \right)$$

$$x_2^{n+1} = \frac{1}{11}\left(x_1^{n+1} + x_3^{n} + x_4^{n} + 25 \right)$$

$$x_3^{n+1} = \frac{1}{10}\left(-2x_1^{n+1} + x_2^{n+1} + x_4^{n} + 11 \right)$$

$$x_4^{n+1} = \frac{1}{8}\left(-3x_2^{n+1} + x_3^{n+1} + 15 \right)$$

start with intial approximation

x^{0} ( ) to get $x^{(1)}$

$X_i = (0,0,0,0)$

$$X_i^{(1)} = (0.6000,\ 2.3272,\ -0.9873,\ 0.8789)$$

Subsequent iteration give the following values.

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

we stopped at iteration no.5 because

$$\frac{\|x^5 - x^4\|_\infty}{\|x^5\|_\infty} = \frac{0.0008}{2} = 4 \times 10^{-4}$$

* Note:
for the same system the use of Jacobi method required 10 iteration to get the same accuracy.

③ Successive over-relaxation method.

If the Gauss-seidel iteration equation are written as.

$$x_1^{n+1} = x_1^{n} + \frac{1}{a_{11}}\left(b_1 - a_{11}x_1^{n} - a_{12}x_2^{n} - a_{13}x_3^{n}\right)$$

$$x_2^{n+1} = x_2^{n} + \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{22}x_2^{n} - a_{23}x_3^{n}\right)$$

$$x_3^{n+1} = x_3^{n} + \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1} - a_{33}x_3^{n}\right)$$

This lead us to the successive over-relaxation method or (SOR) iteration which is defined as:-

$$x_1^{n+1} = x_1^{n} + \frac{\omega}{a_{11}}\left(b_1 - a_{11}x_1^{n} - a_{12}x_2^{n} - a_{13}x_3^{n}\right)$$

$$x_2^{n+1} = x_2^{n} + \frac{\omega}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{22}x_2^{n} - a_{23}x_3^{n}\right)$$

$$x_3^{n+1} = x_3^{n} + \frac{\omega}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1} - a_{33}x_3^{n}\right)$$

where

ω is The acceleration parameter, or Relaxation factor

generally $1 < \omega < 2$

Note:- if $\omega = 1$ leads us to

Gauss-seidel method.

Also the above iteration can be written as

$$x_1^{n+1} = (1-w)\,x_1^{n} + \frac{w}{a_{11}}\left(b_1 - a_{12}x_2^{n} - a_{13}x_3^{n}\right)$$

$$x_2^{n+1} = (1-w)\,x_2^{n} + \frac{w}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{23}x_3^{n}\right)$$

$$x_3^{n+1} = (1-w)\,x_3^{n} + \frac{w}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1}\right)$$

In General form.

$$X_i^{n+1} = (1-\omega)\, x_i^{n} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}\, X^{n+1} - \sum_{j=i+1}^{m} a_{ij}\, X^{n}\right)$$

Example

If the following linear system

$$4x_1 + 3x_2 = 24$$
$$3x_1 + 4x_2 - x_3 = 30$$
$$-x_2 + 4x_3 = -24$$

has The Sol $(3, 4, -5)$

Compare Gauss-Seidel & Successive over-relaxation

Sol Gauss-Seidel

$$x_1^{n+1} = \frac{1}{4}\left(24 - 3x_2^{n}\right)$$

$$x^{n+1} \quad \frac{1}{\ }\left(3\_ - 3x_1^{n+1} + x_3^{n}\right)$$

$$_2 = 4 (30 \qquad 3$$

$$x_3^{n+1} = \frac{1}{4}\left(-24 + x_2^{n+1}\right)$$

and SOR

$$x_1^{n+1} = (1-w)\,x_1^{n} + \frac{w}{4}\left(24 - 3x_2^{n}\right)$$

$$x_2^{n+1} = (1-w)\,x_2^{n} + \frac{w}{4}\left(30 - 3x_1^{n+1} + x_3^{n}\right)$$

$$x_3^{n+1} = (1-w)\,x_3^{n} + \frac{w}{4}\left(-24 + x_2^{n+1}\right)$$

take $w = 1.25$.

Table 7.3

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	5.250000	3.1406250	3.0878906	3.0549316	3.0343323	3.0214577	3.0134110
$x_2^{(k)}$	1	3.812500	3.8828125	3.9267578	3.9542236	3.9713898	3.9821186	3.9888241
$x_3^{(k)}$	1	−5.046875	−5.0292969	−5.0183105	−5.0114441	−5.0071526	−5.0044703	−5.0027940

Table 7.4

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	6.3125000	2.6223145	3.1333027	2.9570512	3.0037211	2.9963276	3.0000498
$x_2^{(k)}$	1	3.5195313	3.9585266	4.0102646	4.0074838	4.0029250	4.0009262	4.0002586
$x_3^{(k)}$	1	−6.6501465	−4.6004238	−5.0966863	−4.9734897	−5.0057135	−4.9982822	−5.0003486